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Quantization of non-standard Hamiltonian systems

Alejandro Corichi† and Michael P Ryan Jr‡

Center for Gravitational Physics and Geometry, Department of Physics, The Pennsylvania State University, University Park, PA 16802, USA

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Abstract. The quantization of classical theories that admit more than one Hamiltonian description is considered. This is done from a geometrical viewpoint, both at the quantization level (geometric quantization) and at the level of the dynamics of the quantum theory. A spin-$\frac{1}{2}$ system is taken as an example in which all the steps can be completed. It is shown that the geometry of the quantum theory imposes restrictions on the physically allowed non-standard quantum theories.

1. Introduction

The problem of quantization of a classical theory is at least 70 yr old, but the term 'quantization' always has had a somewhat loose meaning. There is no such thing as *the* quantization prescription that takes a classical theory and produces the 'correct' quantum theory.

There are three main approaches to canonical quantization: algebraic [1], geometric [2], and group theoretic quantization [3]. They differ, roughly speaking, in the basic structures on phase space that they regard as fundamental in order to construct a quantum theory. In each of these approaches one is led to make several choices along the way that might yield inequivalent quantum theories. Well known examples of these ambiguities are the factor ordering problem and different representations of the CCR in QFT, for example.

The quantization schemes mentioned above have, however, a common feature. They assume that the classical system to be quantized is unique, that is, that there is a preferred classical description for the system. On the other hand, from the classical viewpoint, there might be more than one perfectly valid way of representing a given system. These alternative descriptions are called *non-standard* Hamiltonian systems. The aim of this paper is to explore the possibility of quantization starting from different classical theories.

The programme of quantization of a non-standard Hamiltonian dynamics has its roots in work of Feynman reported by Dyson [4] and its extension by Hojman and Shepley [5]. Feynman's original work showed that Poisson-bracket relations place strong constraints on the types of forces allowed in physical systems. Hojman and Shepley generalized Feynman's work and were able to show that a consistent quantization with a set of commuting coordinates led to a second-order Lagrangian in those coordinates. Hojman then constructed a consistent Poisson-bracket Hamiltonian theory for first-order equations of motion of the form $\dot{x}^i = f^i(x^j)$ [6]. We will discuss this formalism in more detail below.

† E-mail address: corichi@phys.psu.edu

‡ Permanent address: Instituto de Ciencias Nucleares, UNAM, A. Postal 70-543, México 04510 DF, Mexico.

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This programme could be seen as yet another ambiguity in the quantization process or, if viewed from a different perspective, as a new avenue for finding possibly valid quantum theories. This would be the case, for instance, if the given system has more than one classical description without any *a priori* criteria for choosing the 'correct' one.

We will proceed as follows. In the introduction we will recall the basic steps of geometric quantization, pointing out the choices one makes in the process and discussing the possible implications in the final quantum theory. Section 2 reviews the possibility of different classical descriptions or 'non-standard Hamiltonian systems'. We consider as an example the classical spin-$\frac{1}{2}$ particle. Section 3 recalls the geometry of quantum mechanics as proposed by Ashtekar and Schilling [16], focusing on the spin-$\frac{1}{2}$ particle. We have chosen this simple system because it shows the connection between the classical and quantum system very clearly, even though this simplicity means that the system is so constrained that there is very little freedom to allow consistant quantization of non-standard Hamiltonian systems. The basic programme is discussed in section 4 for the spinning particle. The obstructions to quantizing the non-standard description are isolated. Section 5 discusses the results and suggests some directions for further research. Throughout the paper, the 'abstract index notation' is employed. For a discussion on the notation see [7].

1.1. Geometric quantization

By quantization we mean the process of finding a quantum theory from some known classical theory. The starting point for all canonical quantization schemes is a classical system described in terms of simplectic geometry. Let us recall the basic notions in order to set the notation [8, 9].

The *phase space* of the system consists of a manifold Γ of dimension $\dim(\Gamma) = 2n$ (real). Physical states are represented by the points on the manifold. *Observables* are smooth functions on Γ. There is a non-degenerate, closed two-form Ω defined on it. That is, the form Ω_{ab} satisfies $\nabla_{[c}\Omega_{ab]} = 0$, and if $\Omega_{ab}V^b = 0$ then $V^b = 0$. Therefore, an inverse Ω^{ab} exists which defines an isomorphism between the cotangent and the tangent space at each point of Γ. The existence of the *simplectic two-form* Ω endows (Γ, Ω) with a *simplectic structure*.

A vector field V^a generates infinitesimal canonical transformations if its Lie drags the simplectic form, i.e.

$$\mathcal{L}_V \Omega = 0. \tag{1.1}$$

This condition is equivalent to saying that locally it is of the form: $V^b = \Omega^{ba}\nabla_a f := X_f^b$ and it is called the *Hamiltonian vector field of f (w.r.t. Ω)*. Note that the simplectic structure gives us a mapping between functions on Γ and Hamiltonian vector fields. Thus, functions on phase space (i.e. observables) are generators of infinitesimal canonical transformations.

The Lie algebra of vector fields induces a Lie algebra structure on the space of functions.

$$\{f, g\} := \Omega_{ab}X_g^a X_f^b = \Omega^{ab}\nabla_a f \nabla_b g \tag{1.2}$$

such that $X^a_{\{f,g\}} = -[X_f, X_g]^a$.

Since the simplectic form is closed, it can be obtained locally from a *simplectic potential*, ω_a,

$$\Omega_{ab} = 2\nabla_{[a}\omega_{b]}. \tag{1.3}$$

Time evolution is given by a vector field, f^a, whose integral curves are the dynamical trajectories of the system. On phase space there is a *preferred* function, the *Hamiltonian*,

H, whose Hamiltonian vector field corresponds directly to f^a, i.e.

$$f^a = \Omega^{ab}\nabla_b H. \tag{1.4}$$

Adopting the viewpoint that all observables generate canonical transformations we see that the motion generated by the Hamiltonian corresponds to 'time evolution'. The 'change' in time of the observables will be simply given by the Poisson bracket of the observable with H ($\dot{g} = f^a\nabla_a g = \Omega^{ab}\nabla_a g\nabla_b H = \{g, H\}$).

So far, not very much has been assumed about the phase space Γ. It can be any (even-dimensional) manifold with complicated topology, compact, open, etc. The simplectic structure Ω and the function H are assumed to be given *a priori*. Note that they might not be unique. From the classical viewpoint the only 'observable' entities are the dynamical trajectories f^a of the system (the equations of motion). They could have come from more than one pair (Ω, H)†.

However, if the system has a configuration space C, then the phase space is automatically 'chosen' to be the cotangent bundle of the configuration space T^*C. There is also a preferred one-form on T^*C which can be taken to be the simplectic potential which determines uniquely the simplectic structure (the Liouville form). Therefore, the fact that a configuration space exists picks out for us the phase space and the simplectic two-form.

The programme of quantization can be divided into two parts: kinematical and dynamical. The kinematical part deals with the problem of defining a good prescription for going 'from Poisson brackets to commutators' in a consistent way. That is, it should start with the classical system and produce a Hilbert space of states. The dynamical part deals with the Hamiltonian, i.e. the generator of dynamical evolution.

We will concentrate on geometric quantization whose starting point is a simplectic manifold (Γ, Ω). There is no *a priori* assumption about the structure of the phase space Γ. It can be completely general. In particular it can include the case in which Γ is compact, i.e. it is *not* a cotangent bundle.

There are two steps in geometric quantization. The first one involves defining a Hilbert space on the full phase space. Wavefunctions are, roughly speaking, functions on Γ. Any observable can be 'quantized'. The second step involves introducing an additional structure on Γ, a *polarization* that will select those wavefunctions that depend only on 'half of the coordinates'. Physical observables are those that respect, in a way to be defined below, the polarization.

We start with a Hamiltonian system as defined above. We define what are called *prequantum wavefunctions*. They are cross sections, Ψ, of a complex line bundle over Γ. The corresponding $U(1)$ connection is the simplectic potential ω_a whose curvature is the simplectic two form Ω_{ab}. For each trivialization ω_a a function Ψ_ω corresponds. If we change ω by a gauge transformation $\omega_a \to \omega_a + \nabla_a g$ then

$$\Psi_{\omega'} = \mathrm{e}^{\mathrm{i}g/\hbar}\Psi_\omega. \tag{1.5}$$

There is a Hermitian inner product in this complex vector space given by the Liouville measure on Γ. The pre-Hilbert space would be the completion with respect to this inner product.

Any observable f ($f : \Gamma \to R$) has a corresponding symmetric operator, O_f, defined by:

$$O_f\Psi = \frac{\hbar}{\mathrm{i}}X_f^a\nabla_a\Psi + f\Psi := \frac{\hbar}{\mathrm{i}}X_f^a\left(\partial_a - \frac{\mathrm{i}}{\hbar}\omega_a\right)\Psi + f\Psi. \tag{1.6}$$

† There is another, even more drastic, possibility. There could be another f'^a that could have the same integral curves as f^a. Such systems are called S-equivalent [10]. We will not consider them here.

Note that we are now extending the definition of ∇ to 'act' on sections Ψ of the complex line bundle. These operators are: (i) linear; (ii) gauge-covariant, (iii) symmetric (formally self-adjoint).

The assignment $f \to O_f$ is one-to-one and preserves the natural Lie algebra structure,

$$[O_f, O_g] = -\mathrm{i}\hbar O_{\{f,g\}} \tag{1.7}$$

that is, one can assign a consistent operator to all observables.

It is known that 'actual' quantum wavefunctions depend only on 'half' of the variables. We have to 'split' Γ into two parts. This is done by choosing a *polarization*, P. At each point, γ, it assigns a maximal subspace, $P|_\gamma$, of the complexified tangent space such that:

(i) V^a and $W^a \in P|_\gamma$ then $[V, W]^a \in P|_\gamma$ for all γ;

(ii) for all $V^a, W^a \in P$ then $\Omega_{ab}V^aW^b = 0$ for all γ.

If P is real we have a 'real polarization'. The first condition implies that through each point of Γ there passes an n-dimensional submanifold, which is tangent to the subspace $P|_\gamma$. The phase space is then foliated by n-dimensional submanifolds. The second condition implies that the Poisson bracket of any two coordinates of this submanifold vanishes.

Given a polarization, a *quantum wavefunction* is a cross section, Ψ, satisfying

$$V^a\nabla_a\Psi = 0 \tag{1.8}$$

for all $V^a \in P$. This is called the *polarization condition*.

This condition tells us that the wavefunction depends only on n coordinates q^i 'in involution'. (For instance, if we have a configuration space, C, with coordinates q^i, the standard polarization is the 'vertical polarization' spanned by $\{\frac{\partial}{\partial p_i}\}$. We then have that $\{q^i, q^j\} = 0$.)

Classical observables whose pre-quantum operators become well-defined operators are *good observables*. The condition is,

$$[O_f, V^a\nabla_a]\Psi = 0 \tag{1.9}$$

for all $V^a \in P$. This can be written classically as $[X_f, V]^a \in P$ for all V^a ($\mathcal{L}_V X_f \in P$). We say then that X_f^a *preserves the polarization, P*. In particular, the operators corresponding to the coordinates q^i preserve the vertical polarization and are therefore good observables.

A special kind of complex polarization is called Kähler. An almost complex structure is a tensor field $J_a{}^b$ such that $J_a{}^bJ_b{}^c = -\delta_a{}^c$, and it is a canonical transformation: $J_a{}^bJ_c{}^d\Omega_{bd} = \Omega_{ac}$. Then,

$$g_{ab} := \Omega_{ac}J^c{}_b \tag{1.10}$$

is symmetric, non-degenerate, positive definite metric. The triplet (Ω, J, g) equips Γ with an almost Kähler structure. We can construct on the phase space a Hermitian (complex) inner product whose real part is given by g and the imaginary part by Ω, i.e. $(,) = \frac{1}{2}g(,) - \frac{\mathrm{i}}{2}\Omega(,)$.

The tensor field, J, has eigenvectors in the complexified tangent space. Let us decompose any (complexified) V^a into two parts,

$$V^a_\pm := \tfrac{1}{2}(V^a \mp iJ^a{}_bV^b) \tag{1.11}$$

where V^a_+ is an eigenvector of J with eigenvalue i. Let us choose the vector space spanned by those eigenvectors. It is an n-dimensional (complex) vector space, and $\Omega_{ab}V^a_+V^b_+ = 0$. If the distribution is integrable (the manifold can be given as complex charts), the polarization is called Kähler.

In this case the polarization condition, on the section of the Hermitian line bundle, involves considering *holomorphic* sections. When the phase space Γ is compact it is

necessary to have holomorphic sections. This is relevant, for instance, for the quantization of spin systems.

Note that prequantization is a purely kinematical step. It produces a (non-physical) Hilbert space on Γ and every observable is prequantizable. There is no external input (other than the original (Ω, H) pair).

The choice of polarization, on the other hand, has both kinematical and dynamical content. It is kinematical because it singles out the physically relevant quantum states from the prequantum Hilbert space and defines what the physically admissible observables are, namely those that preserve the polarization. This choice also has dynamical implications since the Hamiltonian might *not* be compatible with P. It is the choice of polarization that might lead to inequivalent quantum theories.

2. Non-standard classical theory

As we mentioned in the introduction, we are interested in considering systems that might have a non-standard classical description. By this we mean systems that admit more than one Hamiltonian formulation or systems that obey certain equations of motion that do *not* come from a variational principle.

This section has two parts. In the first we review the non-standard Hamiltonian systems mentioned above, considering a generalization of the simplectic formalism, namely that of Poisson structures on a manifold. The second part takes a spinning classical particle as a particular example of a system that admits non-standard descriptions.

2.1. Poisson structures and non-standard dynamics

In the introduction we gave an overview of the standard Hamiltonian dynamics in terms of a simplectic structure Ω_{ab}. It is possible to define dynamics by introducing a more general structure known as a *Poisson (bracket) structure* [8, 9]. It consists of a bivector $\Pi^{ab} = \Pi^{[ab]}$ on Γ satisfying the Jacobi identity:

$$\Pi^{c[d}\nabla_c \Pi^{ab]} = 0. \tag{2.1}$$

It defines naturally a 'generalized' Poisson bracket between functions on Γ.

$$\{f, g\}_\Pi := \Pi^{ab}\nabla_b f \nabla_a g. \tag{2.2}$$

It also defines a mapping from functions to vector fields

$$\overset{\pi}{X}{}^a_f := \Pi^{ab}\nabla_b f. \tag{2.3}$$

Note that Π^{ab} might be degenerate, in which case there will be *Casimir functions*. For instance, if $\nabla_a C$ is a degenerate 'direction' of Π^{ab} ($\Pi^{ab}\nabla_b C = 0$), then $\{f, C\}_\Pi \equiv 0$, $\forall f$. That is, C 'commutes' with all functions on Γ.

In the case of a non-degenerate simplectic structure, its inverse, Ω^{ab}, defines (locally) an 'almost' one-to-one mapping between functions and Hamiltonian vector field, that is, two functions will define the same vector field if they differ by, at most, a constant function. On the other hand, for a degenerate Poisson structure, given a Casimir function C, then two functions f and g will define the same vector field $X^a_f = \Pi^{ab}\nabla_b f$ if $f = g + h(C)$ where $h(C)$ is *any* (differentiable) function of C.

Given a phase space Γ, the dynamical evolution of a system is given by the integral curves of a vector field V^a. The vector field gives at each point of Γ a set of equations

of motion for the system. If we choose some local coordinates x^μ, $\mu = 1, \ldots 2n$, then the rate of change of each coordinate x^μ is given by the Lie derivative of x^μ along V^a,

$$\dot{x}^\mu := \mathcal{L}_V x^i = V^a \nabla_a (x^\mu) = V^\mu(x). \tag{2.4}$$

Recall that in the x^μ coordinate system, $V^a = V^\mu(x)(\frac{\partial}{\partial x^\mu})^a$.

A natural question is whether the given system of first-order differential equations can be put in a Hamiltonian form. That is, does there exist a Poisson structure Π^{ab} and a function h such that $V^a = \Pi^{ab}\nabla_b h$? If the set of equations came from a (second-order) variational principle, then the Poisson structure is the inverse of the (naturally defined) simplectic structure $\Omega^{(0)}_{ab}$ on $\Gamma = T^*C$ and the Hamiltonian, h, is the Legendre transform of the Lagrangian (for non-singular systems).

There might be, however, *more than one* Poisson structure that makes the equations Hamiltonian, with another Hamiltonian. Those systems are known as *bi*-Hamiltonian [11].

In the case when the set of equations does not come from a variational principle, there is in principle no natural way of putting them in Hamiltonian form. A programme for doing this has recently been proposed by Hojman [6]. The underlying idea is that one should use the symmetries of the equations of motion in order to construct a Poisson structure. Let us summarize the Hojman construction for systems with $N = 2n$ constants of motion C_i, $(N-1)$ of which do not depend explicitly on time. That is, one knows them as explicit functions of the coordinates (a fairly strong requirement, equivalent to knowing the full classical solution). The preceding requirement is sufficient to be able to reduce the equations to Hamiltonian form. It is, of course, not necessary for constructing the Hamiltonian theory.

This Π^{ab} may be constructed by summing elements of the form

$$\Pi^{ab} = \mu(x)\varepsilon^{ab\lambda_1\ldots\lambda_{N-2}}\nabla_{\lambda_1}C_1 \ldots \nabla_{\lambda_{N-2}}C_{N-2} \tag{2.5}$$

where $\varepsilon^{ab\lambda_1\ldots\lambda_{N-2}}$ is the N-index Levi-Civita symbol, and $\mu(x)$ is a function of the coordinates to be explained below. This Π^{ab} satisfies the Jacobi identity. The $C_1, \ldots, C_{N-2}$ are time-independent constants of motion. The Hamiltonian is defined by $H = C_{N-1}$, along with $C_N = t + d_N$, where d_N is time independent. This can always be achieved by a change of coordinates. Hojman has another construction that uses a symmetry of the equations of motion, without needing to know some constants of motion in the explicit form. For more details see [6].

Suppose that for a given set of equations that come from a Lagrangian, we have been able to construct a non-degenerate Π by means of the Hojman procedure. Let us denote by Ω'_{ab} the corresponding two-form ($\Omega'_{ab}\Pi^{bc} = \delta^c_a$). If the Poisson structure Π is compatible with Ω^{ab}†, then there will be a tensor field K^a_b such that

$$\Omega'_{ab} = K^c_a \Omega_{cb}. \tag{2.6}$$

Note that since Ω is invertible, we then have $K^d_a = \Omega'_{ab}\Omega^{bd}$. We will call this mapping a *Hojman transformation*.

2.2. Classical description of a spin-$\frac{1}{2}$ particle

As we mentioned in the introduction, the example we would like to use to describe the change of Poisson structures in quantum mechanics is the simplest quantum system, that of a spin-$\frac{1}{2}$ particle. In order to investigate the relationship between the classical and quantum theories we would like to study the classical problem equivalent to that of a quantum spin-$\frac{1}{2}$

† Two Poisson structures are said to be *compatible* if their sum is also a Poisson structure [11].

particle. The main difficulty with this idea is that, strictly speaking, there is no classical limit to this problem. There are a number of 'classical' limits that have been proposed [12], but we will use a limit in terms of Grassman variables. We would like to find a limit of the quantum theory based on the three spin operators $\hat{S}_i = \overline{h}\sigma_i$, the σ_i the Pauli matrices with Hamiltonian $\hat{H} = A\hat{S}_3$, A = constant. Notice that $\hat{S}_i^2 = \overline{h}^2$, and $[\hat{S}_i, \hat{S}_j] = \overline{h}\varepsilon_{ijk}\hat{S}_k$, and $\{\hat{S}_i, \hat{S}_j\}_+ = 0$, $i \neq j$. As $\overline{h} \to 0$, we get $\hat{S}_i^2 = 0$, $[\hat{S}_i, \hat{S}_j] = 0$ and $\{\hat{S}_i, \hat{S}_j\}_+ = 0$, and there is no set of classical numbers that can obey these relations. If we write the classical variables as $S_i = \varepsilon s_i(t)$, where the s_i are commuting functions of t and ε is a constant Grassman number, then $S_i^2 = 0$ ($\varepsilon^2 = 0$), $[S_i, S_j] = 0 = \{S_i, S_j\}_+$.

Assume we have a Hamiltonian, H, in principle a function of some coordinates q_i, $i = 1, 2, 3$, and $S_i = \beta_{ik}p_k$, where the p_i are the momenta conjugate to the q_i, and $\beta_{ij} = \beta_{ij}(q)$ (the angular velocities are $\omega_i = \alpha_{ij}(q)\dot{q}_j$, where $\alpha_{ij}\beta_{jk} = \delta_{ik}$), then

$$\dot{S}_j + \gamma_{jk\ell}\frac{\partial H}{\partial S_k}S_\ell = 0 \tag{2.7}$$

if H does not depend explicitly on the q_i, i.e. $H = H(S_i)$. For a rigid body $\gamma_{jk\ell} = \alpha_{\ell m}(\frac{\partial \beta_{mk}}{\partial q_n}\beta_{nj} - \frac{\partial \beta_{mj}}{\partial q_n}\beta_{nk}) = -\varepsilon_{jk\ell}$. If we take $H = AS_3$ then

$$\dot{S}_i = \varepsilon_{i3k}AS_k \tag{2.8}$$

or,

$$\varepsilon\dot{s}_i = \varepsilon_{i3k}A\varepsilon s_k \tag{2.9}$$

and

$$\dot{s}_i = \varepsilon_{i3k}As_k. \tag{2.10}$$

These imply that $s_3 = \text{constant} = K_1$ and

$$\dot{s}_1 = -As_2 \tag{2.11}$$

$$\dot{s}_2 = As_1 \tag{2.12}$$

so $s_1^2 + s_2^2 = \text{constant}$. These mean that $s_1^2 + s_2^2 + s_3^2 = \mathcal{S}^2 = \text{constant}$ which implies that the classical-state space is a two-sphere. The system's orbits lie on the two-sphere of radius $\mathcal{S}$ and since s_3 is a constant they are parallels of 'latitude'. If we look at the equations for s_i, $\dot{s}_3 = 0$ and (2.11), (2.12), they can be written as

$$\dot{s}_i = \Pi_{ij}\frac{\partial H}{\partial s_j} \tag{2.13}$$

with $H = As_3$ and $\Pi_{ij} = \varepsilon_{ijk}s_k$.

This is precisely an example of a very well studied system with a Poisson structure. Systems that have rotational degrees of freedom (a rigid body for example), have a common description coming from the fact that the rotation group $SO(3)$ acts on the system, as we now recall [8, 9]. The phase space is given by a three-dimensional vector space (that we can identify with R^3) with coordinates s_i (it is the dual of the Lie algebra $SO(3)$). The Poisson structure is given by

$$\Pi_{ij} = C^k{}_{ij}s_k \tag{2.14}$$

where $C^k{}_{ij} = \delta^{kn}\varepsilon_{nij}$ are the structure constants of $SO(3)$. It is clearly degenerate (any asymmetric tensor field in an odd-dimensional space is). Note, however, that Π_{ij} induces a non-degenerate simplectic structure on each sphere of radius $\mathcal{S}$. R^3 is then foliated by *leaves* of simplectic manifolds. Furthermore, the 'natural' Casimir function is $K_0 = \frac{1}{2}\delta^{ij}s_is_j$

which is clearly constant on each sphere. All Hamiltonian vector fields generated by Π are tangent to the spheres and therefore leave the Casimir unchanged.

Note that Π can be written as

$$\Pi_{ij} = \varepsilon_{nij}\frac{\partial K_0}{\partial s_n} \tag{2.15}$$

which is precisely of the form of (2.5).

A remark is in order. With our formalism we could recover the Euler equations for a rigid body if we choose the Hamiltonian to be the kinetic energy $T = I^{ij}s_i s_j$, where I^{ij} is the inverse of the inertia tensor. The Hamiltonian we have chosen for our system, $H = As_3$, is therefore not the 'kinetic' energy of a rigid body, but resembles more that of a 'point-like' object that might interact with an external potential (a constant magnetic field, for example).

The idea now, in order to find different descriptions for the system, is to use the Hojman prescription for different constants of motion. We have the functions $K_1 = s_3$ and $K_2 = s_1^2 + s_2^2$. Following Hojman [13] we can now take $C = C(K_1, K_2)$, any arbitrary function of (K_1, K_2), and a new 'Hamiltonian' $H = H(K_1, K_2)$, also any function of K_1 and K_2, and define

$$\tilde{\Pi}_{ij} = \mu(s_\ell)\varepsilon_{ijk}\frac{\partial C}{\partial s_k}. \tag{2.16}$$

We would then like to have the equations of motion for s_i as

$$\dot{s}_i = \tilde{\Pi}_{ij}\frac{\partial H}{\partial s_j}. \tag{2.17}$$

We can have the same equations as before if we choose μ properly and C and H satisfy one condition. If we look at the s_3 equation we have

$$\begin{aligned}\dot{s}_3 = {} & \mu\left[\frac{\partial C}{\partial K_1}\frac{\partial K_1}{\partial s_2} + \frac{\partial C}{\partial K_2}\frac{\partial K_2}{\partial s_2}\right]\left[\frac{\partial H}{\partial K_1}\frac{\partial K_1}{\partial s_1} + \frac{\partial H}{\partial K_2}\frac{\partial K_2}{\partial s_1}\right] \\ & -\mu\left[\frac{\partial C}{\partial K_1}\frac{\partial K_1}{\partial s_1} + \frac{\partial C}{\partial K_2}\frac{\partial K_2}{\partial s_1}\right]\left[\frac{\partial H}{\partial K_1}\frac{\partial K_1}{\partial s_2} + \frac{\partial H}{\partial K_2}\frac{\partial K_2}{\partial s_2}\right]\end{aligned} \tag{2.18}$$

and since K_1 does not depend on s_1 or s_2,

$$\dot{s}_3 = -2\mu s_1 s_2\left[\frac{\partial C}{\partial K_2}\frac{\partial H}{\partial K_2} - \frac{\partial C}{\partial K_2}\frac{\partial H}{\partial K_2}\right] = 0. \tag{2.19}$$

For s_1

$$\begin{aligned}\dot{s}_1 = {} & \mu\left[\frac{\partial C}{\partial K_1}\frac{\partial K_1}{\partial s_3} + \frac{\partial C}{\partial K_2}\frac{\partial K_2}{\partial s_3}\right]\left[\frac{\partial H}{\partial K_1}\frac{\partial K_1}{\partial s_2} + \frac{\partial H}{\partial K_2}\frac{\partial K_2}{\partial s_2}\right] \\ & -\mu\left[\frac{\partial C}{\partial K_1}\frac{\partial K_1}{\partial s_2} + \frac{\partial C}{\partial K_2}\frac{\partial K_2}{\partial s_2}\right]\left[\frac{\partial H}{\partial K_1}\frac{\partial K_1}{\partial s_3} + \frac{\partial H}{\partial K_2}\frac{\partial K_2}{\partial s_3}\right] \\ = {} & 2\mu s_2\left[\frac{\partial C}{\partial K_1}\frac{\partial H}{\partial K_2} - \frac{\partial C}{\partial K_2}\frac{\partial H}{\partial K_1}\right].\end{aligned} \tag{2.20}$$

We can achieve $\dot{s}_1 = -As_2$ if $\Delta \equiv \frac{\partial C}{\partial K_1}\frac{\partial H}{\partial K_2} - \frac{\partial C}{\partial K_2}\frac{\partial H}{\partial K_1} \neq 0$ and we take $\mu = -\frac{A}{2\Delta}$. It is easy to show that his choice of μ also gives $\dot{s}_2 = As_1$, so we recover the original equations of motion.

As an example of this procedure, take the normal Hamiltonian $H = As_3$ and $C = s_1^2 + s_2^2$. If we look at the plane $s_1 = 0$, the orbits intersect the circle $s_3^2 + s_2^2 = 1$. The lines of constant $s_3 = H/A$ and C are perpendicular straight lines that form a coordinate grid over

the half plane given by the s_2s_3-plane with $s_2 > 0$. The sphere $s_1^2 + s_2^2 + s_3^2 = \mathcal{S}^2$ intersects this half plane in a semicircle, and any point on this semicircle represents the initial point of a possible orbit, and if we rotate the semicircle around the s_3-axis then a point on it traces out a parallel of 'latitude'. In the rectangular grid of C and H/A we can always specify this point by particular values of C and H/A.

Now, the equation for s_i is

$$\frac{\mathrm{d}s_i}{\mathrm{d}t} = \mu(s_\ell)\varepsilon_{ijk}\frac{\partial C}{\partial s_k}\frac{\partial H}{\partial s_j}. \tag{2.21}$$

Note that this has the form

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \mu(\boldsymbol{s})(\boldsymbol{\nabla}H) \times (\boldsymbol{\nabla}C) \tag{2.22}$$

where $\boldsymbol{\nabla}C$ and $\boldsymbol{\nabla}H$ are the two-dimensional gradients of C and H which are the the normals to the coordinate curves. We have $\boldsymbol{\nabla}H \times \boldsymbol{\nabla}C = |\boldsymbol{\nabla}H \times \boldsymbol{\nabla}C|e_1$, where e_1 is the unit vector in the s_1 direction. Since in the $s_1 = 0$ plane

$$\frac{\mathrm{d}s_1}{\mathrm{d}t} = -As_2 \tag{2.23}$$

we see that (2.22) gives this if we take $\mu = -As_2/|\boldsymbol{\nabla}H \times \boldsymbol{\nabla}C|$. From [13] we see that this μ works for all s_1, s_2.

As long as they form a complete coordinate grid in the s_2s_3-plane, any functions C and H can be used in the formulation. Note that if $\boldsymbol{\nabla}H$ is parallel to $\boldsymbol{\nabla}C$ at any point (or the norm of one of the vectors is zero), μ blows up. This is the condition in [13] for the non-existence of μ. Notice also that H is no longer the energy.

Let us now try to understand what we are doing from a geometrical viewpoint. The fact that we are using a preferred function (the Casimir) to define the Poisson structure means that one-forms w_a 'transverse' to the $C =$ constant surfaces are precisely the degenerate directions of Π. Hamiltonian vector fields are always tangent to the surfaces and therefore the motion they generate lies within them. In the standard case of the rigid body, for example, the surfaces on which the Casimir is constant are spheres precisely because they are the orbits of the rotation group (coadjoint action on the dual of the Lie algebra) on R^3. The simplectic structure induced on the spheres from the Poisson structure on R^3 is precisely ($1/\mathcal{S}$ times) the area element (recall that any non-degenerate two-form on a surface is proportional, by means of a conformal factor, to the area element).

Suppose that we now define a new Poisson structure via a function whose surfaces of constant value are not spheres but some 'ellipsoids' (with rotational symmetry around the s_3-axis). Now, the surfaces will not be the orbits of the rotation group in three dimensions (see [13] for a particular choice in which the resulting deformed algebra is $SU(2)_q$). The change in the induced simplectic structure, the 'Hojman transformation', will be a simple conformal transformation. We can conclude then that by a rescaling of the simplectic structure and a corresponding change in the Hamiltonian, we have an infinite number of classical descriptions for the system.

As we mentioned above, we would now like to apply the idea of changing the simplectic structure to quantum mechanics. In the next section we will discuss this formulation and its extension to 'Kähler quantum mechanics' in the context of the spin-$\frac{1}{2}$ example outlined above. We will see that two obstructions to doing this in the most simple-minded way exist. These are both related to the fact that we need to define a probability structure on the quantum-mechanical phase space. Probability structures are often given in terms of linear operators on a Hilbert space. We will see that both the definition of probabilities in

'Kähler quantum mechanics' and the realization of dynamical quantities as linear operators place constraints on the possible simplectic structures that are allowed. Since the system is basically two-dimensional, we will see that these constraints are so strong that there is no way to change the simplectic structure without changing the quantum physics of the system. Whether this is a feature of higher-dimensional systems is an open question.

3. Quantum mechanics

The problem we want to address in this paper is the possible quantization of systems that admit non-standard descriptions. If the system admits more that one classical description, we are led to ask whether the quantum theories are equivalent. If not, what are the criteria to choose the 'correct' classical description?

As we mentioned in the introduction, there are, roughly speaking, two different sets of issues concerning quantum mechanics that one has to address: kinematical and dynamical. The kinematical conditions, so to speak, that the constructed quantum theory should satisfy, are mainly related to the Heisenberg uncertainty relations. Commuting quantum observables can, in principle, be simultaneously measured. Such quantum observables correspond to classical observables that have vanishing Poisson brackets among them. Therefore, there is in principle a way of distinguishing between, for instance, two different Poisson structures. If the Poisson structure in the classical theory is degenerate, there will be Casimir functions and, therefore, corresponding quantum Casimir operators. This will lead to 'super-selected' sectors that should be detected experimentally.

There is another set of issues that one has to consider when analysing the dynamical content of the theory. Quantum mechanics is a theory of measurement. If the theory is to pass the test of 'validity', it should provide probabilities for measuring eigenvalues of various operators as functions in time, that should be compatible with measurements. This is a condition to be satisfied by the dynamical evolution of the quantum system. This condition is analogous to the corresponding classical condition that the dynamical evolution should be the integral curves of a preferred vector field. This 'dynamical condition' has a very clean geometrical formulation when quantum mechanics is cast in geometric language.

3.1. Geometry of quantum mechanics

Quantum mechanics, with all its postulates, can be put into geometric language. In this section we will recall the geometry of quantum mechanics. For details see [14–18].

The description we will give is for systems with a finite-dimensional Hilbert space but the generalization to the infinite-dimensional case is straightforward [17]. Denote by $\mathcal{P}$ the space of rays in the Hilbert space $\mathcal{H}$. In this case $\mathcal{P}$ will be the complex projective space $\mathbb{C}P^n$, since $\mathcal{H}$ can be identified with $\mathbb{C}^n$.

It is convenient to view $\mathcal{H}$ as a *real* vector space equipped with a complex structure (recall that a complex structure J is a linear mapping $J : \mathcal{H} \to \mathcal{H}$ such that $J^2 = -1$). Let us decompose the Hermitian inner product into real and imaginary parts,

$$\langle \Psi | \Phi \rangle = G(\Psi, \Phi) - \mathrm{i}\Omega(\Psi, \Phi) \tag{3.1}$$

where G is a Riemannian inner product on $\mathcal{H}$ and Ω is a simplectic form.

Let us restrict our attention to the sphere, S, of normalized states. The true space of states is given by the quotient of S by the $U(1)$ action of states that differ by a 'phase', i.e. the projective space $\mathcal{P}$. The complex structure J is the generator of the $U(1)$ action (J plays the role of the imaginary unit i when the Hilbert space is taken to be real). Since

the phase rotations preserve the norm of the states, both the real and imaginary parts of the inner product can be projected down to $\mathcal{P}$.

Therefore, the structure on $\mathcal{P}$ which is induced by the Hermitian inner product is given by a Riemannian metric g and a simplectic two-form $\mathbf{\Omega}$. The pair $(g, \mathbf{\Omega})$ defines a Kähler structure on $\mathcal{P}$ (recall that a Kähler structure is a triplet $(M, g, \mathbf{\Omega})$ where M is a complex manifold (with complex structure J), g is a Riemannian metric and $\mathbf{\Omega}$ is a simplectic two-form, such that they are compatible).

The space $\mathcal{P}$ of quantum states then has the structure of a Kähler manifold, so, in particular, it is a simplectic manifold and can be regarded as a 'phase space' by itself. It turns out that the quantum dynamics can be described by a 'classical dynamics', that is, with the same simplectic description that is used for classical mechanics. Let us see how it works. In quantum mechanics, Hermitian operators on $\mathcal{H}$ are generators of unitary transformations (through exponentiation) whereas in classical mechanics, generators of canonical transformations are real-valued functions $f : \mathcal{P} \to \mathbb{R}$. We would then like to associate with each operator F on $\mathcal{H}$ a function f on $\mathcal{P}$. There is a natural candidate for such a function: $f := \langle F \rangle|_S$ (denote it by $f = \langle F \rangle$). The Hamiltonian vector field X_f of such a function is a Killing field of the Riemannian metric g. The converse also holds, so there is a one-to-one correspondence between self-adjoint operators on $\mathcal{H}$ and real-valued functions ('quantum observables') on $\mathcal{P}$ whose Hamiltonian vector fields are symmetries of the Kähler structure.

There is also a simple relation between a natural vector field on $\mathcal{H}$ generated by F and the Hamiltonian vector field associated with f on $\mathcal{P}$. Consider on S a 'point' ψ and an operator F on $\mathcal{H}$. Define the vector $X_F|_\psi := \frac{\mathrm{d}}{\mathrm{d}t} \exp[-JFt]\psi|_{t=0} = -JF\psi$. This is the generator of a one parameter family (labelled by t) of unitary transformation on $\mathcal{H}$. Therefore, it preserves the Hermitian inner product. The key result is that X_F projects down to $\mathcal{P}$ and the projection is precisely the Hamiltonian vector field X_f of f on the simplectic manifold $(\mathcal{P}, \mathbf{\Omega})$.

Dynamical evolution is generated by the Hamiltonian vector field X_h when we choose as our observable the Hamiltonian $h = \langle H \rangle$. Thus, Schrödinger evolution is described by Hamiltonian dynamics, exactly as in classical mechanics.

One can define the Poisson bracket between a pair of observables (f, g) from the inverse of the simplectic two-form $\mathbf{\Omega}^{ab}$,

$$\{f, g\} := \mathbf{\Omega}(X_g, X_f) = \mathbf{\Omega}^{ab}(\partial_a f)(\partial_b g). \tag{3.2}$$

The Poisson bracket is well defined for arbitrary functions on $\mathcal{P}$, but when restricted to observables, we have,

$$\langle -\mathrm{i}[F, G] \rangle = \{f, g\}. \tag{3.3}$$

This is in fact a slight generalization of Ehrenfest's theorem, since when we consider the 'time evolution' of the observable f we have the Poisson bracket $\{f, h\} = \dot{f}$,

$$\dot{f} = \langle -\mathrm{i}[F, H] \rangle. \tag{3.4}$$

We have seen that the simplectic aspect of the quantum state space is completely analogous to classical mechanics. Notice that, since only those functions whose Hamiltonian vector fields preserve the metric are regarded as 'quantum observables' on $\mathcal{P}$, they represent a very small subset of the set of functions on $\mathcal{P}$.

There is another facet of the quantum state space $\mathcal{P}$ that is absent in classical mechanics: Riemannian geometry. Roughly speaking, the information contained in the metric g is concerned with those features which are unique to the quantum description, namely, those

related to measurement and 'probabilities'. We can define a Riemannian product (f, g) between two observables as

$$(f, g) := g(X_f, X_g) = g^{ab}(\partial_a f)(\partial_b g). \tag{3.5}$$

This product has a very direct physical interpretation in terms of the dispersion of the operator in the given state:

$$(f, f) = 2(\Delta F)^2. \tag{3.6}$$

Therefore, the length of X_f is the uncertainty of the observable F.

The metric g also has an important role in those issues related to measurements. Note that eigenvectors of the Hermitian operator F associated with the quantum observable f correspond to points ϕ_i in $\mathcal{P}$ at which f has local extrema. These points correspond to zeros of the Hamiltonian vector field X_f, and the eigenvalues f_i are the values of the observable $f_i = f(\phi_i)$ at these points.

If the system is in the state Ψ, what are the probabilities of measuring the eigenvalues f_i? The answer is strikingly simple: measure the geodesic distance given by g from the point Ψ to the point ϕ_i (denote it by $d(\Psi, \phi_i)$). The probability of measuring f_i is then,

$$P_i(\Psi) = \cos^2[d(\Psi, \phi_i)]. \tag{3.7}$$

Therefore, a state Ψ is more likely to 'collapse' to a nearby state than to a distant one when a measurement is performed. For a geometric approach to the 'reduction of the state vector' see [19]. We will now turn our attention to spin systems and in particular the quantum theory of a spin-$\frac{1}{2}$ particle.

3.2. The spin-$\frac{1}{2}$ system

In this part we will find the quantum theory of a spin-$\frac{1}{2}$ particle starting from the classical description of section 2. We will then discuss the quantum theory in the geometric language just described.

3.2.1. Geometric quantization of spin systems. In section 2, we arrived at a kinematical description for systems with 'rotational degrees of freedom', that includes spin systems. We saw that the physically relevant space is $\mathbb{R}^3$ that is foliated by spheres of radius $\mathcal{S}$. That is, for each value of $\mathcal{S}$ we have a sphere which corresponds to the reduced phase space of a particle with classical 'intrinsic angular momentum' equal to $\mathcal{S}$. Since each sphere is a simplectic manifold with a perfectly defined simplectic structure on it, we can employ the machinery of geometric quantization that was outlined in the introduction.

We then have, $\Gamma = S^2$, $\Omega_{ab} = \mathcal{S} \sin\theta \nabla_{[a}\phi\nabla_{b]}\theta$, where we have chosen spherical coordinates (θ, ϕ) for the sphere. Note that the simplectic two-form is $1/\mathcal{S}$ times the area element of a sphere of radius $\mathcal{S}$.

The first step in geometric quantization is to construct the prequantum line bundle. There are, however, some integrality conditions that must be satisfied so that the prequantum line bundle exists. These conditions are the generalizations of the Bohr–Sommerfeld quantum conditions:

$$\frac{1}{2\pi\hbar}\int_{S^2} \Omega = k \tag{3.8}$$

where k is an integer. Since $\int_{S^2} \Omega = 4\pi\mathcal{S}$, the condition reads $\mathcal{S} = \frac{\hbar}{2}k$. This is precisely the quantization of spin! What this condition tells us is that the only simplectic manifolds that

can be quantized are those that correspond to classical systems whose angular momentum is an integer multiple of $\frac{\hbar}{2}$.

The next step is to find a polarization in the phase space Γ. Note that the sphere S^2 is a compact manifold and therefore does not correspond to a cotangent bundle. Luckily the sphere is a complex manifold and therefore admits a Kähler structure. We can coordinatize it by z (recall that the Riemann sphere is the complex plane with the point at infinity). The simplectic two form then reads,

$$\Omega = \mathrm{i}k\hbar \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + z\bar{z})^2}. \tag{3.9}$$

The Hilbert space of states will correspond then to holomorphic sections of a complex line bundle over the sphere. A standard theorem in complex analysis shows that the space of such sections is *finite*-dimensional. Furthermore, holomorphic sections can be represented as functions on the coordinate z as follows,

$$\Psi(z) = \sum_{l=0}^{k} \binom{k}{l} \psi_l z^l \tag{3.10}$$

where ψ_l are constants. In this way, one gets all the finite-dimensional, unitary, irreducible representations of $SU(2)$.

Since we are interested in the spin-$\frac{1}{2}$ representation, we have to consider the $k = 1$ case, that is, the 'smallest' quantizable sphere. The Hilbert space in this case is given by elements of the form,

$$\Psi = \psi_0 + \psi_1 z. \tag{3.11}$$

Each element of the Hilbert space, $\mathcal{H}$, will then be characterized by two complex numbers. We have recovered the standard $SU(2)$ two-component spinors. The inner product is then,

$$\langle \Phi | \Psi \rangle = \tfrac{1}{2}(\bar{\phi}_0 \psi_0 + \bar{\phi}_1 \psi_1). \tag{3.12}$$

For details see [2].

3.2.2. Geometry of a quantum spin-$\frac{1}{2}$ system. The spin degrees of freedom of a spin-$\frac{1}{2}$ particle provide a very clear example of the geometric structures described in section 2.1. In this case the Hilbert space, $\mathcal{H}$, is formed by vectors on $\mathbb{C}^2$: $\binom{\alpha}{\beta}$ where α and β are complex numbers. As we saw above, it is convenient then to consider $\mathcal{H}$ as a real vector space. Instead of a column vector in $\mathbb{C}^2$ we will have column vectors on R^4:

$$\Psi = \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix} \tag{3.13}$$

where a, b, c, e are real numbers.

The Hermitian inner product $\langle \Psi | \Phi \rangle$ between $\Psi = \binom{\alpha}{\beta}$ and $\Phi = \binom{\gamma}{\delta}$ given by

$$\langle \Psi | \Phi \rangle = \bar{\alpha}\gamma + \bar{\beta}\delta \tag{3.14}$$

induces a metric G and a simplectic two-form Ω on R^4:

$$\begin{aligned} G_{ij} &= [\nabla_i a \nabla_j a + \nabla_i b \nabla_j b + \nabla_i c \nabla_j c + \nabla_i e \nabla_j e] \\ \Omega_{ij} &= 2(\nabla_{[i} a \nabla_{j]} b + \nabla_{[i} c \nabla_{j]} e). \end{aligned} \tag{3.15}$$

Normalized states then satisfy,

$$\langle \Phi | \Phi \rangle = a^2 + b^2 + c^2 + e^2 = 1. \tag{3.16}$$

Thus, the space S corresponds to the three-sphere S^3.

We know that the quantum space of states $\mathcal{P}$ will be the projection of S^3 under the action of the $U(1)$ action. That is, S has the structure of a principal fibre bundle with fibre S^1 and base space $\mathcal{P} = S^2$:

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \pi \big\downarrow \\ & & S^2. \end{array} \tag{3.17}$$

This corresponds precisely to one of the Hopf bundles over the two-sphere S^2.

In order to show the projection π explicitly and to recover common coordinates on the sphere S^2 we introduce the coordinates (α, β, δ) on S^3 as follows,

$$\begin{aligned} a &= \cos\left(\frac{\beta}{2}\right)\cos(\delta+\alpha) \\ b &= \cos\left(\frac{\beta}{2}\right)\sin(\delta+\alpha) \\ c &= \sin\left(\frac{\beta}{2}\right)\cos(\delta-\alpha) \\ e &= \sin\left(\frac{\beta}{2}\right)\sin(\delta-\alpha). \end{aligned} \tag{3.18}$$

It is straightforward to compute the induced simplectic structure on S:

$$\bar{\Omega}_{ij} = \sin\beta \nabla_{[i}\alpha\nabla_{j]}\beta. \tag{3.19}$$

It is clear that the degenerate direction of $\bar{\Omega}$ is $(\frac{\partial}{\partial\delta})^j$, which is precisely the direction of the 'phase change' generated by J.

The induced metric on S is

$$\bar{G}_{ij} = \nabla_i(\alpha)\nabla_j(\alpha) + \tfrac{1}{4}\nabla_i(\beta)\nabla_j(\beta) + \nabla_i(\delta)\nabla_j(\delta) - 2\cos\beta\nabla_{(i}(\alpha)\nabla_{j)}(\delta). \tag{3.20}$$

It is clear that $\bar{\Omega}$ corresponds to the pullback of $\mathbf{\Omega}$ under π ($\bar{\Omega} = \pi^*\mathbf{\Omega}$). We can find the metric defined in the orbits of the degenerate direction, and define $(g, \mathbf{\Omega})$ on $\mathcal{P} = S^2$ with ordinary spherical coordinates ($\theta = \beta, \phi = 2\alpha$) to be

$$\mathbf{\Omega}_{ab} = \tfrac{1}{2}\sin\theta\nabla_{[a}\phi\nabla_{b]}\theta \tag{3.21}$$

$$g_{ab} = \tfrac{1}{4}[\sin^2(\theta)\nabla_a(\phi)\nabla_b(\phi) + \nabla_a(\theta)\nabla_b(\theta)]. \tag{3.22}$$

Quantum observables correspond on $\mathcal{H}$ to Hermitian 2×2 matrices. A basis for those matrices is given by the Pauli matrices. They are associated with the generators of rotations in three dimensions and are the 'angular momentum' operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$, satisfying ordinary commutation relations: $[\hat{S}_i, \hat{S}_j] = \mathrm{i}\overline{h}\varepsilon_{ijk}\hat{S}_k$. We know that there are three functions on $\mathcal{P}$ which correspond to the 'observables' in the 'quantum phase space';

$$\begin{aligned} x &:= \langle\hat{S}_x\rangle = \overline{h}(ac+be) = \frac{\overline{h}}{2}\sin\theta\cos\phi \\ y &:= \langle\hat{S}_y\rangle = \overline{h}(ae-cb) = \frac{\overline{h}}{2}\sin\theta\sin\phi \\ z &:= \langle\hat{S}_z\rangle = \frac{\overline{h}}{2}[(a^2+b^2)-(c^2+e^2)] = \frac{\overline{h}}{2}\cos\theta. \end{aligned} \tag{3.23}$$

It is a curious fact that they are also the Cartesian coordinates of a sphere of radius $\overline{h}/2$.

Let us now consider dynamical evolution. Without a loss of generality we can take the Hamiltonian to be $H = A\hat{S}_z$. The corresponding observable on $\mathcal{P}$ is $h = \langle \hat{H} \rangle = A\frac{\hbar}{2}\cos\theta$. Given h and $\mathbf{\Omega}$ we can compute the equations of motion for the coordinates (θ, ϕ):

$$\begin{aligned} \dot{\theta} &= \mathbf{\Omega}^{ab}\partial_a\theta\partial_b h = 0 \\ \dot{\phi} &= \mathbf{\Omega}^{ab}\partial_a\phi\partial_b h = -A\hbar. \end{aligned} \tag{3.24}$$

That is, the quantum evolution is given by a 'point' travelling on S^2 at constant 'latitude' θ and with constant angular velocity $\dot{\phi} = -A\hbar$.

Note that the quantum description in terms of 'Kähler geometry' for the spin-$\frac{1}{2}$ particle coincides exactly with the classical description given in section 2. for the chosen Hamiltonian. The spheres in both cases have, however, very different origins. In one case it is the smallest quantizable *reduced phase space*. In the quantum case it is the *projective* 'quantum phase space' coming from the Hilbert space of states. For an alternative treatment of the spin-$\frac{1}{2}$ with complex coordinates see [18].

4. Non-standard quantum Hamiltonian systems

Notice that our previous discussion means that it is possible to describe the quantization of a system in two stages. In order to see this, it is simpler to think of these stages in reverse, that is, as one method of constructing a classical theory from a known quantum theory. In this 'classicalization' one would begin with a Hilbert space, $\mathcal{H}$, and a set of observables given as linear operators on $\mathcal{H}$. We could now project onto the space of rays $\mathcal{P}$, which, since it is a phase space itself and observables are now represented by real-valued functions, the system is represented by a 'classical theory' with at least a large part (if not all) of the content of the quantum theory defined on the Hilbert space. The main addition to this 'classical' theory is the probability structure given by (3.7) based on the Riemannian metric g_{ab}. If one were able to ignore the probability structure of this simplectic manifold, one could think of quantum mechanics on $\mathcal{P}$ as nothing more than another classical theory. Our programme of 'classicalization' would then simply be a map from $\mathcal{P}$ to another simplectic manifold Γ, the phase space of the usual classical theory. This mapping is in general 'many-to-one' since the space $\mathcal{P}$ is much larger than Γ. If it is only a small subset of $\mathcal{P}$, the *coherent states*, that will behave in a 'semiclassical' fashion. We can represent the process by the following diagram,

$$\begin{array}{ccc} \mathcal{H} & & \\ \downarrow & & \\ \mathcal{P} & \longrightarrow & \Gamma. \end{array} \tag{4.1}$$

The usual process of 'quantization' is to leap from Γ directly to $\mathcal{H}$, but one might just try to reverse the direction of the arrows in (4.1), first constructing the 'Kähler quantum theory' on $\mathcal{P}$, then 'raising' the observables on $\mathcal{P}$ to Hermitian operators on $\mathcal{H}$. Notice that it could be possible to stop this procedure at $\mathcal{P}$ if one could be certain that *all* the properties of quantum mechanics (such as the superposition of states) could be realized in terms of observables on $\mathcal{P}$ and the probability structure generated by g_{ab}.

The programme we are addressing in this paper involves, however, the ordinary quantization process from Γ to $\mathcal{H}$ and then considering the 'projected' geometrical formulation on $\mathcal{P}$. The classical theory we are starting with, having a modified simplectic geometry defined on it, will yield a different geometry on $\mathcal{P}$. That is, the simplectic structure $\mathbf{\Omega}$ on $\mathcal{P}$ will have some information of the corresponding one on Γ. The question we are

led to ask is: Is the 'non-standard' geometry induced on the constructed quantum theory compatible with experiment?

From now on we will restrict our attention to the simplest system at hand: the spin-$\frac{1}{2}$ particle, and show explicitly that there are obstructions to this procedure at each level. Given that the various Hamiltonian descriptions for the classical system differ by only a conformal transformation, the set of issues we will be addressing are the ones we called 'dynamical' in the discussion at the beginning of section 3. We shall encounter two difficulties in the programme. First, while we will see that it is quite simple to mirror the change of simplectic structure given by (2.15) and recover the dynamics of the quantum system on $\mathcal{P}$ (in the sense of recovering the integral curves of the original system), but we will find that it is more difficult to maintain the probability structure in terms of g_{ab} that does not exist in the purely classical system. Secondly, we will see that realizing the dynamics of the non-standard Hamiltonian system in terms of a linear Hamiltonian operator is impossible in most cases.

We would like to change the simplectic two-form on $\mathcal{P}$ for the spin-$\frac{1}{2}$ system and find a new Hamiltonian function, $\tilde{h}$, which gives the same set of integral curves that are given in section 3. We must also require that the *physical* predictions are the same in terms of measurement. Recall that the probability of measuring the eigenvalue o_i of an operator $\hat{O}$ when the system is in state Ψ is given by the geodesic distance from Ψ to the point Φ_i ($\hat{O}\Phi_i = o_i\Phi_i$): $P(\Psi, o_i) = \cos^2[d(\Psi, \Phi_i)]$. This implies that in order to recover the same physical predictions, not only the dynamical trajectory must be the same but also the geodesic distance to the eigenstates.

Let us consider a double Stern–Gerlach experiment in which we first measure $\hat{S}_z$ and then look only at the particles that had a spin 'up'. In our picture, this corresponds to considering a quantum state located at the 'north pole' ($\theta = 0$). We now make a second measurement with a new measuring device. The spatial orientation of the new apparatus corresponds precisely to the orientation of the eigenstates (which lie on 'antipodal points') on the sphere. The probability of measuring spin 'up' and 'down' will depend only on the angle along maximal circles, from the north pole to the 'podes'. Since the system is rotationally symmetric, we can rotate both detectors while keeping their relative orientation fixed and the probabilities will not change. That operation corresponds to 'fixing the 'up' direction of the detectors' in (x, y, z) space and rotating the sphere. Since the distance along the sphere must be the same, we conclude that the metric on S^2 should be rotational symmetric, which is a property of the metric inherited from the Hermitian inner product. Let us denote by $\overset{o}{g}_{ab}$, the metric defined by equation (3.22) ($\overset{o}{g}_{ab} = \frac{1}{4}[\sin^2(\theta)\nabla_a(\phi)\nabla_b(\phi) + \nabla_a(\theta)\nabla_b(\theta)]$).

We can conclude then that the metric g should be equal to $\overset{o}{g}$, together with the integral curves. The question that we are led to ask is: can we find a new $\tilde{\Omega}$ and $\tilde{h}$ such that the Hamiltonian vector fields of $\tilde{h}$ and g_{ab} are the same? Since any two-form on S^2 is given by a conformal transformation from the 'canonical' two-form $\mathbf{\Omega}$ defined by equation (3.21), what we are looking for is precisely the conformal factor μ in section 2, such that,

$$\tilde{\Omega}^{ab} = \mu\mathbf{\Omega}^{ab}. \tag{4.2}$$

It is easy to see that we can find a $\tilde{h}$ such that the dynamical evolution is the same. The condition, in the (θ, ϕ) coordinates, is

$$\begin{pmatrix} 0 \\ -A\overline{h} \end{pmatrix} = \begin{pmatrix} 0 & \tilde{\Omega}^{\theta\phi} \\ -\tilde{\Omega}^{\theta\phi} & 0 \end{pmatrix} \begin{pmatrix} \partial_\theta \tilde{h} \\ \partial_\phi \tilde{h} \end{pmatrix}. \tag{4.3}$$

This set implies that $\partial_\phi \tilde{h} = 0$, or, $\tilde{h} = f(\theta)$, so the system reduces to one equation:

$$A\overline{h} = \tilde{\Omega}^{\theta\phi} f' \tag{4.4}$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}\theta}$.

Therefore, $\tilde{\Omega}^{\theta\phi} = A\overline{h}\frac{1}{f'}$. To solve the system, we could fix f and then define $\tilde{\Omega}$ from the previous equation. This would give us the conformal factor as $\mu = -\overline{h}A\frac{\sin\theta}{f'}$.

However, recall that $\mathcal{P}$ must have a Kähler structure, so g and Ω must be compatible in the sense that $g_{ab} = J_a^c \Omega_{cb}$. Can we change Ω arbitrarily and still have a compatible system for fixed g? The answer to this question is, as expected, in the negative. This is because in order to have a Kähler structure, the function f' has to be $f' = K\sin\theta$. That is, if and only if $\mu = C$, where K and C are real and constant.

We have to conclude that for the spin-$\frac{1}{2}$ system it is impossible to have a non-standard quantum Hamiltonian dynamics compatible with observation: there is no freedom to change Ω and h.

The second obstruction when changing the simplectic structure in quantum mechanics is that we would normally like to have the 'Kähler quantum mechanics' on $\mathcal{P}$ come from a system of operators in a Hilbert space whose expectation values on $\mathcal{P}$ would generate the observables. If we attempt to do this for $\tilde{h}$, and even if we were to ignore the negative result above, we are still restricted by the fact that $\tilde{h}$ must be a function of only θ. Even if we try to let $\tilde{h}$ be any function of θ, in this simple case if $\tilde{h}$ is to be the image of a linear Hermitian operator on the space of vectors in $\mathcal{H}$, the operator $\hat{\tilde{H}}$ must be of the form

$$\hat{\tilde{H}} = \zeta I + \tfrac{\eta}{2}\hat{S}_x + \tfrac{\kappa}{2}\hat{S}_y + \tfrac{\lambda}{2}\hat{S}_z \tag{4.5}$$

with ζ, η, κ, λ real. This means that

$$\begin{aligned}\tilde{h} &= \zeta + \frac{\eta}{2}\langle\hat{S}_x\rangle + \frac{\kappa}{2}\langle\hat{S}_y\rangle + \frac{\lambda}{2}\langle\hat{S}_z\rangle \\ &= \zeta + \eta\frac{\overline{h}}{4}\sin\theta\cos\phi + \kappa\frac{\overline{h}}{4}\sin\theta\sin\phi + \lambda\frac{\overline{h}}{4}\cos\theta\end{aligned} \tag{4.6}$$

must be a function of θ. The only way to satisfy this for all ϕ is to take $\eta = \kappa = 0$. This means that the only possible $\tilde{h}$ that come from linear Hermitian operators are

$$\tilde{h} = Kh + D \tag{4.7}$$

where K and D are real constants. In this case the new μ is $\mu = (1/K)\mu_0$. All other choices of μ must lead to $\hat{\tilde{H}}$, a nonlinear operator.

These two obstructions are very strongly related, since the only way to have well-defined vector fields on a Kähler manifold is if they respect the complex structure (holomorphic vector fields), and in turn, those vector fields correspond to Hamiltonian vector fields coming from linear operators on $\mathcal{H}$ [17].

5. Conclusions and suggestions for further research

We have attempted to transfer to quantum theory an idea originally due to Hojman, that perhaps the usual simplectic structure of classical mechanics is too restrictive, and it might be possible to generalize it. In classical mechanics this is certainly the case, and it may lead to new approaches to solving old problems, and can be used to construct Hamiltonian theories for systems that have no variational principle, and thus no Hamiltonian in the usual sense. We have considered this idea from the viewpoint of changing the simplectic structure

and Hamiltonian of a system that does have a Hamiltonian. Classically this can be done without a loss of generality, since we can easily generate the same solution curves for the system for a large class of simplectic structures.

In the present article, we have investigated the simplest possible quantum system in order to try to isolate sources of problems in quantizing non-standard Hamiltonian systems. It is perhaps not surprising that we were able to achieve the same evolution of the quantum states on $\mathcal{P}$ for any of our changes of Hamiltonian and Poisson structure, since this evolution is nothing more than 'classical' Hamiltonian motion. The extra rigidity of a probability structure was what caused our programme to fail for the simple spin-$\frac{1}{2}$ system. This provides us with at least a hint that the basic problem in all systems will be in the probability structure given by the Riemannian metric. This, too, is not surprising, since changing the Poisson structure is related to changing the uncertainty relations for different observables of the theory. These are in turn related to the 'probabilities' of finding the system in certain states. We cannot be certain that the spin-$\frac{1}{2}$ system has all of the properties of higher-dimensional systems, but it seems reasonable, from the discussion above, that we will always be able to reproduce that evolution of the quantum states, but will encounter difficulties with probability structures.

Our spin-$\frac{1}{2}$ system is also special in the sense that the space $\mathcal{P}$ and Γ are both spheres. This is so because *every* state on $\mathcal{P}$ is a coherent state, and therefore exhibits an almost classical behaviour [20]. For systems of higher dimensions than the spin-$\frac{1}{2}$ particle, either we will have enough freedom to make changes in the simplectic structure without spoiling the probability structure neccesary for quantum mechanics, or the programme will fail as it did for our simple system. Given the manner in which the programme failed for the spin-$\frac{1}{2}$ particle, we conjecture that a similar failure will occur in more complicated systems. If this is true we will have to use more general constructions than Kähler spaces to describe the quantum system.

In our simple example, changing Ω_{ab} on $\mathcal{P}$ leads to a disastrous change in the metric g_{ab} on $\mathcal{P}$ that defines the probability. If it were possible to change Ω_{ab} without changing g_{ab}, we would have a simple solution to the problem. The difficulty here is equation (1.10),

$$g_{ab} = \Omega_{ac} J^c_b \tag{5.1}$$

which relates Ω_{ab} to g_{ab} through the complex structure tensor J^a_b. Note that the complex structure is required to be compatible with G_{ab} and obey $J^b_a J^c_b = -\delta^c_a$. If we make a similarity transformation (such as a coordinate transformation) on J, $J^a_b = S^a_c J^c_d (S^{-1})^d_b$, $J^b_a J^c_b = -\delta^c_a$ is preserved. If one makes such a transformation, both Ω_{ab} and g_{ab} change as 'covariant tensors', which is perfectly acceptable. Notice that if we were to make a more complicated transformation, such as a conformal transformation, on Ω_{ab}, $\Omega_{ab} \to \varphi\Omega_{ab}$, and at the same time insist that g_{ab} remain unchanged in order to preserve the probability structure, we would have to allow $J^a_b \to (1/\varphi)J^a_b$, and $J^b_a J^c_b = -(1/\varphi)^2\delta^c_a$, which is negative definite and non-singular as long as φ is finite and non-zero, but does not obey the defining equation of a complex structure tensor.

In higher-dimensional phase spaces the Hojman transformation $\Omega_{ab} \to K^c_a \Omega_{cb}$ would imply that to maintain the metric g_{ab} invariant one would have to take $J^a_b \to J'^c_b = J^a_c (K^{-1})^c_b$, and, in principle, since the Hojman transformation contains the conformal factor μ, we might expect that $J'^a_b J'^b_c$ would not be equal to $-\delta^a_c$, just as for a conformal transformation. In that case, it would be necessary to postulate 'pseudocomplex structures', where $J^a_b J^b_c$ would be negative definite multiples of δ^a_c but not necessarily $-\delta^a_c$, in order to preserve g_{ab} on changing Ω_{ab}. Note, however, that while the Hojman transformation for a two-dimensional phase space reduces to a pure conformal transformation, the more

general transformation allowed in higher-dimensional phase spaces may still allow us to write $J'^a_b J'^b_c = -\delta^a_c$, in which case J'^a_b is nothing more than a 'deformed complex structure', and this concept has been studied for some time [21]. It is necessary to investigate whether the Hojman transformation allows $J'^a_b J'^b_c = -\delta^a_c$ or not. Another possibility would be the investigation of complex manifolds that admit more than one simplectic structure, the *hyperKähler manifolds*.

Another construction that would permit changing the simplectic structure without deforming the complex structure would be to allow the appropriate transformation on g_{ab} that would preserve J^a_b (in the spin-$\frac{1}{2}$ case a conformal transformation) and define probabilities in some 'conformally invariant' fashion. We will not attempt to consider this idea further here.

One remark is in order. The phase space of the system we started with, namely the sphere S^2, is somewhat special. Perhaps the most notorious property is that it is a *compact* manifold. As a consequence, the Hilbert space in the quantum theory is *finite*-dimensional. Furthermore, it has recently been shown that the *only* classical observables that can be quantized in a way that the prescription $\{,\} \to i\hbar[,]$ is satisfied exactly, are the generators of rotations s_i [22]. This is the equivalent, for S^2, of the Groenewold–Van Hove theorem[23]. Our result for the spin-$\frac{1}{2}$ system is therefore another indication of the 'rigidity' of the structures one can define on the sphere. This has to be contrasted with higher-dimensional (non-compact) phase spaces for which the quantum theory is much richer (infinite-dimensional Hilbert space). In this case one has in fact an infinite number of possible complex structures (this freedom is similar to the one that leads to different inequivalent representations of the CCR in QFT). In this case, the non-standard quantum theory has to satisfy the 'kinematical' requirements related to the Heisenberg uncertainty principle, and possible super-selected sectors, in order to be considered 'valid'. A complete study of the most general case is therefore, still open.

Finally, note that if it were possible to be sure that all of the content of quantum mechanics could be achieved in terms of the evolution and structure of points in $\mathcal{P}$, we would not need to worry about the fact that the time evolution of states, for example, is a reflection of evolution in the Hilbert space, $\mathcal{H}$, that is generated by a nonlinear Hamiltonian operator. If this is not so, then we would be forced to consider the possibility of nonlinear evolution in quantum mechanics, an idea that has been proposed by several authors [18, 24], but one should be justifiably reluctant to propose such a drastic modification to, at the very least, a one-particle model.

Our investigation of the spin-$\frac{1}{2}$ system has given us a glimpse of possible obstructions to doing quantum mechanics based on non-standard Hamiltonian dynamics. While our low-dimensional example may not have enough freedom to allow different simplectic structures, we feel that the type of failure observed there will be a feature of higher-dimensional systems. This, of course, must be studied. If the feature is a general feature, more complicated constructions must be considered. All of these problems will be considered in future articles[25].

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